Announcements

1. Quiz 1 posted
2. No class on Monday 6/28
3. Problem Set 3 will post later today,
   due 7/2
4. Group Work 2 will post Tuesday 6/29
   due 7/2
5. Updated syllabus on Canvas
   (Topic schedule modified)

2.3 Modeling with 1st-order DEs

1. Construct the model
2. Analysis of the model
3. Comparison with Experiment or Observation

Ex Lake Shasta in northern California is the state's largest reservoir and it is formed by the Shasta Dam on the Sacramento River. Create a model for the volume of water in Lake Shasta.

Let $V(t)$ be the volume of water

$r_1(t)$ be the inflow

$r_2(t)$ be the outflow

Then

$$\frac{dV}{dt} = r_1(t) - r_2(t)$$

How can we model $r_1(t)$?

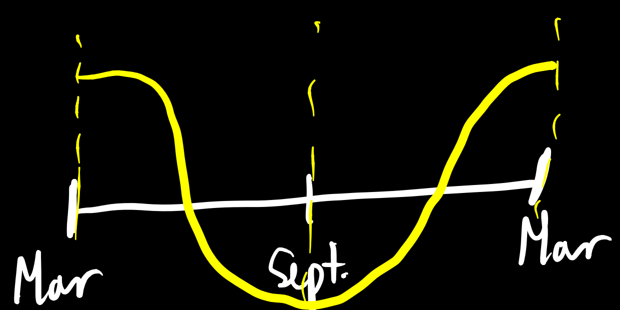
- rain fall
- snow melt

Jan Mar Jun Sept. Dec.

max flow

min flow

Flow



Let the time scale be years so $t=1$ is 1 year. Then

$$r_1(t) = \cos(2\pi t)$$

gives us the desired shape. Note that $t=1$ is March of year 1, $t=2$ is March of year 2, ...

Next, we don't want negative flow, so let's add 1 to $r_1(t)$,

$$r_1(t) = \cos(2\pi t) + 1.$$

How to model $r_2(t)$?

Let's let $r_2(t) = c_1$ for now. That is, there is a constant flow of water passing through the dam.

Drawbacks of this assumption?

- Could empty lake if r_2 is too large
- Lake could overflow if r_2 is too small

What about initial conditions?

The Lake began storing water in 1944, so if we let $t=0$ correspond to March 1944, then $V(0)=0$.

Thus, we have the IVP

$$\frac{dV}{dt} = \cos(2\pi t) + 1 - c_1 \quad , \quad V(0) = 0$$

$$\int \frac{dV}{dt}\, dt = \int \cos(2\pi t) + 1 - c_1 \, dt$$

$$V(t) = \frac{1}{2\pi}\sin(2\pi t) + t - c_1 t + C$$

Now, from IC

$$0 = V(0) = C$$

Therefore,

$$V(t) = \frac{1}{2\pi}\sin(2\pi t) + t - c_1 t$$

We can find suitable values for c_1 by considering V_{max} and V_{min}. That is, we do not want the Lake to overflow or go below a certain volume.

§2.4 Linear vs Nonlinear DEs

<u>Thm</u> Existence and Uniqueness for 1st order Linear Eqns.

If p and g are continuous on the interval $I: \alpha < t < \beta$ and $t_0 \in I$, then there is a unique function $y = \phi(t)$ that satisfies the DE

$$y' + p(t)y = g(t)$$

for each $t \in I$ and satisfies the IC $y(t_0) = y_0$.

"Proof" We found that

$$\mu(t)\, y = \int \mu(t) g(t)\, dt + C$$

$$\mu(t) = e^{\int p(t)\, dt}$$

Since p is continuous for $\alpha < t < \beta$, μ is defined, differentiable, and nonzero. μ and g are continuous, so μg is integrable and $\int \mu g\, dt$ is differentiable. Therefore,

$$y = \frac{1}{\mu(t)}\left(\int \mu(t) g(t)\, dt + C\right)$$

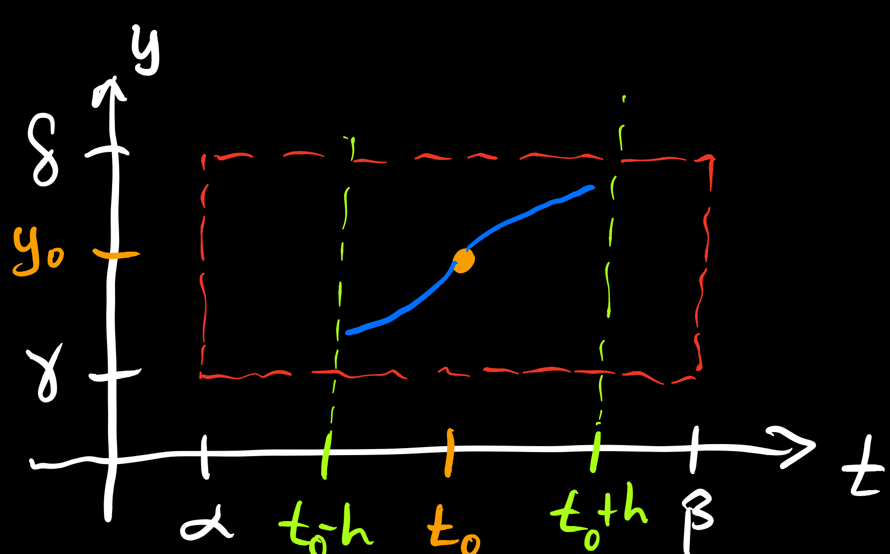
exists and is differentiable for $\alpha < t < \beta$.

Thm Existence and Uniqueness for 1st Order Nonlinear Eqns

Let f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution to the IVP

$$y' = f(t, y), \quad y(t_0) = y_0 .$$

Picture:



Note: The conditions in the theorem are sufficient, but not necessary. That is, we can make a weaker assumption on f.

We can guarantee existence (but not uniqueness) by only assuming f is continuous.

Ex Consider the IVP

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0$$

for $t \geq 0$.

We have $f(t,y) = y^{\frac{1}{3}}$ and $\frac{\partial f}{\partial y} = \frac{1}{3} y^{-\frac{2}{3}}$.

f is continuous everywhere but $\frac{\partial f}{\partial y}$ does not exist when $y = 0$.

Therefore, the previous theorem does not apply, but f is continuous so a solution exists but it is not necessarily unique.

Let's solve the IVP.

$$y^{-\frac{1}{3}} y' = 1$$

$$\frac{3}{2} y^{\frac{2}{3}} = t + c$$

$$y = \left(\frac{2}{3}(t+c)\right)^{3/2}$$

Using IC,

$$0 = y(0) = \left(\frac{2}{3} c\right)^{3/2} \Rightarrow c = 0$$

Therefore,

$$y = \phi_1(t) = \left(\frac{2}{3} t\right)^{3/2}, \quad t \geq 0$$

but also

$$y = \phi_2(t) = -\left(\tfrac{2}{3}t\right)^{3/2}, \quad t \geq 0$$

is a solution to the IVP.

$\phi_2(0) = 0$ IC ✓

$$\phi_2' = -\tfrac{3}{2}\left(\tfrac{2}{3}t\right)^{\frac{1}{2}}\tfrac{2}{3}$$

$$= -\left(\tfrac{2}{3}t\right)^{1/2}$$

$$\phi_2' - (\phi_2)^{\frac{1}{3}} = -\left(\tfrac{2}{3}t\right)^{\frac{1}{2}} - \left(\left(-\tfrac{2}{3}t\right)^{3/2}\right)^{\frac{1}{3}}$$

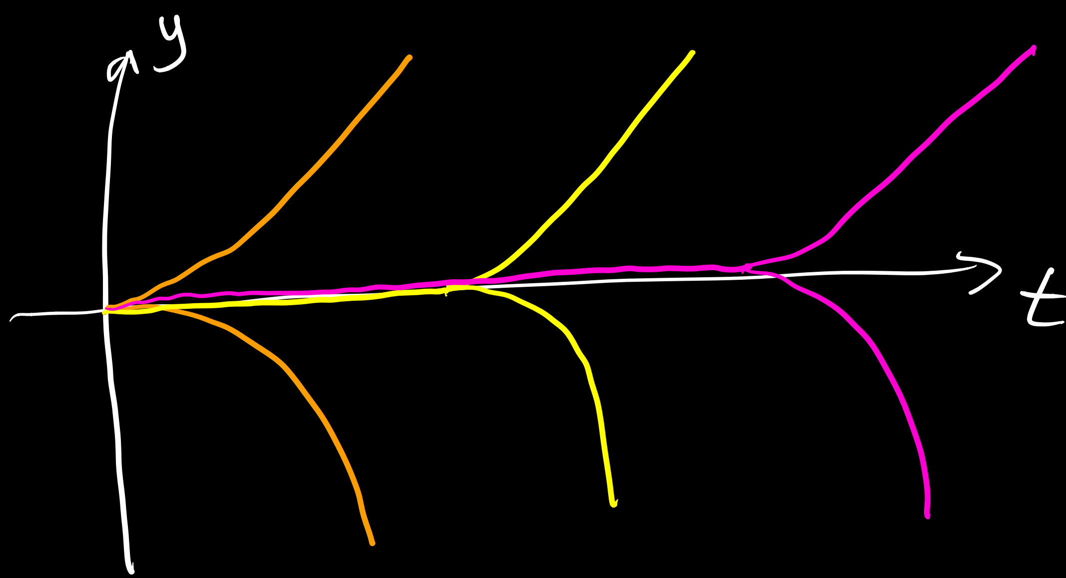
$$= -\left(\tfrac{2}{3}t\right)^{\frac{1}{2}} + \left(\tfrac{2}{3}t\right)^{\frac{1}{2}}$$

$$= 0 \quad ✓$$

For arbitrary t_0

$$y = \begin{cases} 0 & 0 \leq t < t_0 \\ \pm\left(\tfrac{2}{3}(t-t_0)\right)^{3/2} & t \geq t_0 \end{cases}$$

are all solutions to the IVP



Interval of Existence

linear equation : The solution exists in an interval about $t = t_0$ in which p and g are continuous.

nonlinear equation : $y = \phi(t)$ exists as long as $(t, \phi(t))$ remains in the hypothesis region, but $\phi(t)$

is not known ahead of time.

General Solution

linear equation: We can obtain all possible solutions by specifying the constant of integration

nonlinear equation: There may be solutions that cannot be obtained by giving values to the constant.

Implicit Solutions

linear equation: $y = \phi(t)$, explicitly

nonlinear equation: $F(t, y) = 0$, implicit

§ 2.5 Autonomous DEs, Population Dynamics

A first order ODE that does not have the independent variable appear explicitly is called autonomous and is of the form

$$\frac{dy}{dt} = f(y).$$

This equation is separable, and we will examine this equation in the context of population dynamics.

Exponential Growth

Suppose the rate of change of a population is proportional to the current population, then

$$\frac{dy}{dt} = r y$$

growth rate; population

If $y(0) = y_0$, then we get $y = y_0 e^{rt}$

$r > 0 \Rightarrow y \to \infty$ as $t \to \infty$

$r < 0 \Rightarrow y \to 0$ as $t \to \infty$

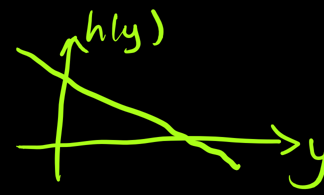
$r = 0 \Rightarrow y = y_0$ for all t

Logistic Growth

Suppose the growth rate depends on the population. Then we have

$$\frac{dy}{dt} = h(y)\, y$$

Consider $h(y) = r - ay$, $a > 0$



$$\frac{dy}{dt} = (r - ay)\,y$$

or

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \quad \text{where } K = \frac{r}{a}$$

r ↑ intrinsic growth rate

This is called the logistic equation.

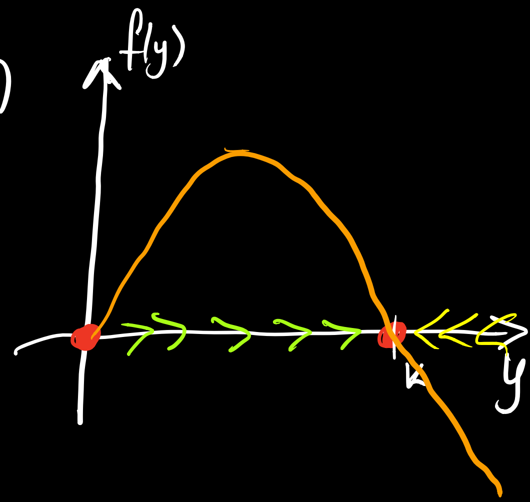
Qualitative Analysis

- Identify equilibrium solutions

$$\frac{dy}{dt} = 0 \quad \text{i.e.} \quad f(y) = 0$$

The zeros of y are called critical points.

Ex $\dfrac{dy}{dt} = r\left(1-\dfrac{y}{K}\right)y = f(y)$



$$0 = r\left(1-\frac{y}{K}\right)y$$

$$y = K \ , \ y = 0$$

For $y \in (0, K)$, $\dfrac{y}{K} < 1$, so $1-\dfrac{y}{K} > 0$

Therefore $f(y) > 0$.

For $y \in (K, \infty)$, $\dfrac{y}{K} > 1$, so $1-\dfrac{y}{K} < 0$

Therefore $f(y) < 0$.

The y-axis is called the phase line.

Note: Previously, we saw plots of y vs t